convection near a corner of two vertical porous plates embedded in porous medium, *Lett. Heat Mass Transfer* **7**, 457–463 (1980).
6. Y. Liu and A. C. Guerra, Free convection in a porous medium near the corner of arbitrary angle formed by two vertical plates, *Int. Commun. Heat Mass Transfer* **12**, 431–440 (1985).
7. Y. Liu, C. Y. Lam and A. C. Guerra, Free convection near a corner formed by two vertical plates embedded in porous medium, *Int. Commun. Heat Mass Transfer* **14**, 125–136 (1987).
8. D. Angirasa and R. L. Mahajan, Natural convection from L-shaped corners with adiabatic and cold isothermal horizontal walls *J. Heat Transfer* **115**, 149–157 (1993).
9. P. J. Roache, *Computational Fluid Dynamics* (Rev. Edn). Hermosa, Albuquerque, NM (1982).
10. W. H. Press, B. P. Flannery, S. A. Teukolsky and W. T. Vetterling, *Numerical Recipes*. Cambridge University Press, New York (1986).
11. P. Cheng and C. T. Hsu, Higher-order approximations for Darcian free convective flow about a semi-infinite vertical flat plate. *J. Heat Transfer* **106**, 143–151 (1984).

Pergamon

Int. J. Heat Mass Transfer. Vol. 40, No. 2, pp. 490–492, 1997

0017–9310/97 $15.00+0.00

PII: S0017–9310(96)00093–2

Symbolic mathematics for the calculation of thermal efficiencies and tip temperatures in annular fins of uniform thickness

ANTONIO CAMPO and R. EUGENE STUFFLE
College of Engineering, Idaho State University, Pocatello, ID 83209, U.S.A.

(*Received* 1 *November* 1995 *and in final form* 29 *February* 1996)

INTRODUCTION

The first heat transfer analyses of fins with different shapes date back to the pioneering contributions of Harper and Brown [1] and Gardner [2], who led the way to numerous research papers on this topic throughout the subsequent years.

Independent of the shape of the fin, the need to estimate the tip temperature as a function of geometric, flow and thermal parameters stems from the fact that these local temperatures have to be within certain bounds to ensure the safety of technical personnel in plant environments (see Muir *et al.* [3]).

Straight fins of uniform thickness are characterized by two dimensions only, length and thickness. Calculation of the heat transfer and the tip temperature of these fins is rather simple, because it requires the evaluation of a hyperbolic tangent and a hyperbolic cosine, respectively. Unfortunately, other fin configurations of equal importance, such as the annular fin of uniform thickness, are characterized by three dimensions, length, thickness and radii ratio. Additionally, the expressions for the heat transfer and the tip temperature involve the evaluation of Bessel functions. These aspects were cleverly circumvented by Gardner [2] who devised a graphical representation of the heat transfer, constructing the so-called fin efficiency diagram. This diagram contained the fin efficiency in the ordinate and a dimensionless thermogeometric parameter as the abscissa. The family of curves was parameterized by the radii ratio, whose range typically goes from 1 up to 5, inclusive. As expected, whenever the radii ratio is not an integer number, the reading in the fin efficiency diagram necessitates 'visual interpolation'. It should be remembered that in those days engineers performed calculations with slide rules. Unequivocally, the above-described computational procedure is incompatible with modern computer aided design (CAD) of annular fins of uniform thickness.

On the other hand, Gardner [2] left untouched an easy parallel procedure linked to the calculation of the corresponding tip temperatures of annular fins of uniform thickness. These calculations could have provided a companion tip temperature diagram to the fin efficiency diagram.

In order to alleviate the minor deficiencies pertinent to the analysis and design of annular fins of uniform thickness, this technical note presents a simple and versatile way that facilitates the rapid determination of both quantities: the fin efficiencies and the tip temperatures, both in terms of the controlling parameters. These calculations have been accomplished with the help of symbolic computational mathematics and the computed results have been post-processed, analyzed and reported in terms of correlation equations of compact form.

MATHEMATICAL ANALYSIS

Consider the steady-state heat transfer in an annular fin of uniform thickness $2t$ in which the inner and outer radius are r_1 and r_2, respectively. For a situation of constant physical properties of the material and negligible heat loss through the tip, the temperature distribution, $\theta(z)$, can be taken from the textbook by Mills [4]

$$\theta(z) = \frac{K_1(z_2)I_0(z)+I_1(z_2)K_0(z)}{I_0(z_1)K_1(z_2)+I_1(z_2)K_0(z_1)}. \quad (1)$$

NOMENCLATURE

h	average heat transfer coefficient
$I_v(\cdot)$	modified Bessel function of the first kind of order v
$K_v(\cdot)$	modified Bessel function of the second kind of order v
k	thermal conductivity
L	fin length, r_2-r_1
Q_i	ideal heat transfer rate
Q_t	heat transfer rate
r	radial coordinate
r_1	inner radius of fin
r_2	outer radius of fin
r_r	radii ratio, r_2/r_1
T	temperature
T_b	base temperature
T_{tip}	tip temperature
T_∞	ambient temperature
$2t$	fin thickness
z	transformed radial coordinate, βr.

Greek symbols

β	thermogeometric parameter, $(h/kt)^{1/2}$
β'	dimensionless β, βL
θ	dimensionless T, $(T-T_\infty)/(T_b-T_\infty)$
η_f	fin efficiency.

Here, θ is the dimensionless temperature, $z = \beta r$ is a coordinate transformation, $\beta = (h/kt)^{1/2}$ is the thermogeometric parameter, and $I_v(\cdot)$ and $K_v(\cdot)$ are the modified Bessel function of the first and second kind of order v, respectively.

Certainly, having access to symbolic mathematics codes, the evaluation of Bessel functions may be performed rapidly and accurately leading to the determination of the temperature distributions along the annular fin. Supplementary to this, there are two thermal quantities of paramount importance for thermal design engineers.

(a) *Heat transfer from the fin*, Q_t

This quantity of global nature may be quantified from the temperature distribution, $\theta(z)$, either by differentiating or by integrating equation (1). Regardless of the approach employed, the final expression becomes

$$Q_t = 4\pi r_1\sqrt{hkt}(T_b-T_\infty)\times\frac{K_1(z_1)I_1(z_2)-I_1(z_1)K_1(z_2)}{I_0(z_1)K_1(z_2)+I_1(z_2)K_0(z_1)}. \quad (2)$$

Next, the fin efficiency, defined as $\eta_f = Q_t/Q_i$, may be conveniently written as

$$\eta_f = \frac{2}{(1+r_r)\beta'}\frac{K_1(z_1)I_1(z_2)-I_1(z_1)K_1(z_2)}{I_0(z_1)K_1(z_2)+I_1(z_2)K_0(z_1)}, \quad (3)$$

where Q_i designates the ideal heat transfer rate from a fin possessing identical characteristics. The customary format for the presentation of the numerical results for the heat transfer rate, Q_t, was devised by Gardner [2] and necessitates the plotting of the fin efficiency, η_f, in the ordinate vs the dimensionless thermogeometric parameter, β', in the abscissa having the radii ratio, r_r, as a parameter in the family of curves.

(b) *Tip temperature of the fin*, $\theta_{tip} = \theta(z_2)$

Despite the fact that the tip temperature (a minimum local temperature), $\theta(z_2)$, seems to be unimportant within the realm of thermal design *per se*, knowledge of its magnitude is vital to prevent the technical personnel working plants receiving burns [3]. Surprisingly, a review of the heat transfer literature reflects that a companion diagram for the dimensionless tip temperature, similar in form to the fin efficiency diagram, has never been reported.

Upon evaluating the temperature at the fin extreme, $\theta(z_2)$, with symbolic computational algebra yields the dimensionless tip temperature

$$\theta_{tip} = \theta(z_2) = \frac{K_1(z_2)I_0(z_2)+I_1(z_2)K_0(z_2)}{I_0(z_1)K_1(z_2)+I_1(z_2)K_0(z_1)} \quad (4)$$

that obviously depends on the dimensionless thermogeometric parameter, β', and the radii ratio, r_r, like in the relation for the fin efficiency.

CORRELATION EQUATIONS

Once representative results for the fin efficiencies and the dimensionless tip temperatures have been computed for realistic combinations of the controlling parameters β' and r_r, the collected data have been subsequently treated using specialized software for multivariate regression analysis. Here, the principal goal was to develop two concise correlation equations, one for the fin efficiencies, η_f, and the other for the dimensionless tip temperatures, θ_{tip}, that can be used by thermal design engineers for the computer aided design of annular fins of uniform thickness. Coincidentally, the resulting correlation equations have identical form and are expressed in an abbreviated format as follows:

$$\eta_f, \theta_{tip} = \frac{1+b\beta'+c\ln(r_r)}{1+d\beta'+e\beta'^2+f\beta'^3+g\ln(r_r)}. \quad (5)$$

Here, the correlation coefficients were very close to one and the numerical values of the coefficients are listed in Table 1.

Ultimately, for purposes of visualization, the correlation equations generated the surfaces drawn in Figs. 1 and 2, which share the same upper and lower bounds for the radii ratio, r_r, utilized for the presentation of a fin efficiency diagram.

CONCLUSIONS

This technical note has shown that after the fin efficiencies and tip temperatures of annular fins of uniform thickness have been determined with symbolic mathematics, it was very easy to apply multivariate regression analysis to the data in order to generate compact correlation equations for the predictions of these two thermal quantities. This idea for having correlation equations may be conceived as a natural extension of the analysis for straight fins, where simple analytical expressions do exist for the evaluation of these two thermal quantities in terms of a single variable, the dimensionless thermogeometric parameter. The combined computational procedure can be easily extended to cases wherein the heat loss through the tip of annular fins has to be taken into account.

The reader will have an enlarged perspective of some issues that are of concern to communities of thermal design engineers and engineering students as well.

Table 1. Coefficients for the fin efficiency and the dimensionless tip temperature

	b	c	d	e	f	g
η_f	-0.025	-0.361	0.045	0.298	-0.035	-0.359
θ_{tip}	-0.126	-0.203	-0.029	0.310	0.094	-0.190

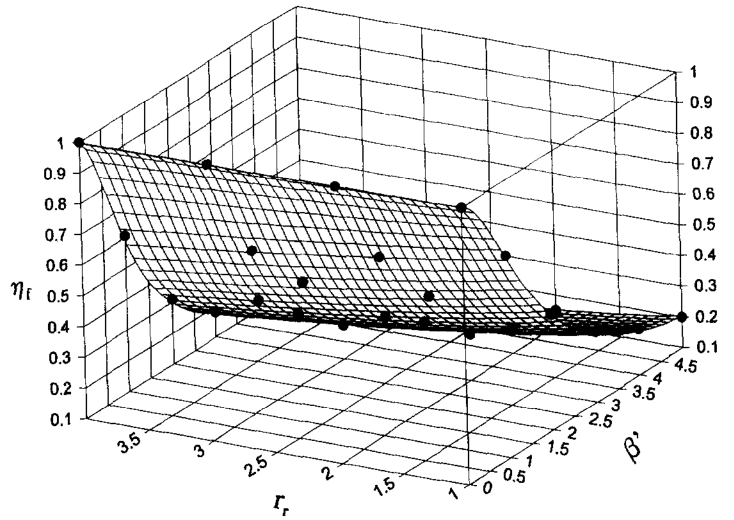

Fig. 1. Variation of the fin efficiency with the parameter β' and the radii ratio r_r.

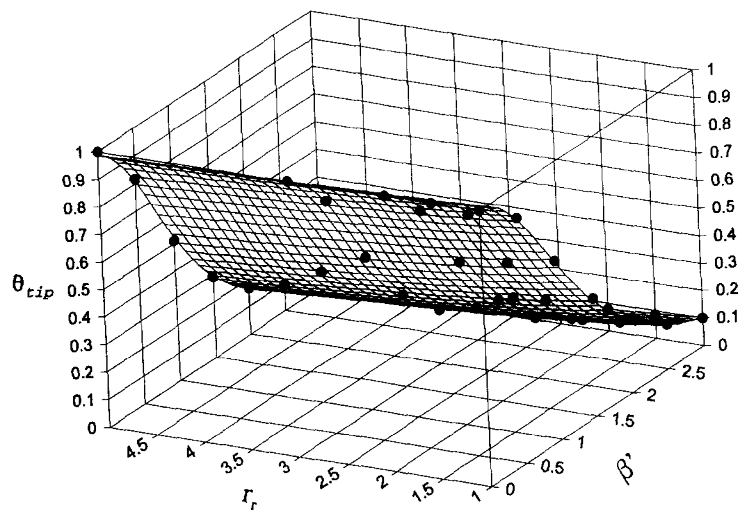

Fig. 2. Variation of the tip temperature with the parameter β' and the radii ratio r_r.

REFERENCES

1. W. P. Harper and D. R. Brown, Mathematical equations for heat conduction in the fins of air-cooled engines, NACA Report 158 (1922).
2. K. A. Gardner, Efficiency of extended surfaces, *Trans. ASME* **67**, 621–631 (1945).
3. I. F. K. Muir, T. L. Barclay and J. A. D. Settle, *Burns and their Treatment*. Butterworths, Boston, MA (1987).
4. A. F. Mills, *Heat Transfer*. Richard Irwin, Boston, MA (1992).